An application of quadratic functions to Australian Government policy on funding schools[*]

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I n the *Sydney Morning Herald* of 23 March 2005, Ross Gittins argued that the funding arrangements for private schools positively encourage parents to move their children from the state system. The then Federal Minister for Education, Dr Brendan Nelson, in a letter to the *Herald* of 25–27 March, responded by saying that 68% of all school pupils go to state schools, and those students receive 76% of Government funds allocated to the totality of all pupils attending schools. He stated also that the policy of subsidising pupils who went to a private school resulted in taxpayer savings of $4 billion. However, the Minister's response did not address the extent to which more money could possibly be saved by having a different subsidy from the one currently offered by the Government.

There are two conflicting factors in offering subsides to private school pupils. On the one hand, the greater the subsidy per pupil, the more pupils will enrol in private schools. On the other hand, the greater the subsidy per pupil the less money will be saved each time a pupil enrols in a private school. How do these factors balance out, and where would an optimal subsidy occur? The problem is closely related to other problems of optimisation that arise in business, industry and public policy. Mathematically, the problem can be modelled by means of a quadratic function that describes how the savings change as the subsidy changes.

Calculation of the current subsidy

Let *m* pupils go to state schools and *n* pupils go to private schools. Then, the proportion of pupils going to state schools is θ, where

$$\theta = \frac{m}{m+n} \tag{1}$$

Assume that for each state school pupil, the Government will pay an amount *a*. Assume also that for each private school pupil, the Government will subsidise that pupil at an amount *s*. The proportion of Government funds spent on the state school system, out of the total of all Government funds spent on both state and private schools is ϕ, where

$$\phi = \frac{ma}{ma+ns} \tag{2}$$

A routine calculation using (1) and (2) gives

$$\frac{s}{a} = \frac{-1+\dfrac{1}{\phi}}{-1+\dfrac{1}{\theta}} \tag{3}$$

The Minister's figures in the *Herald* (25–27 March) give:

$$\theta = \frac{17}{25} \text{ and } \phi = \frac{19}{25}$$

Based on this and allowing for round-off in the Minister's figures, we have from (3) that

$$s = \frac{2}{3}a \tag{4}$$

The mathematical analysis of savings

Since the Government pays an amount a for each state school pupil and pays a subsidy s for each private school pupil, for each private pupil, the Government saves an amount $a - s$. At the subsidy s, let $g(s)$ denote the number of pupils who enrol in private schools. The function g is taken to be *increasing* for, if a greater subsidy is offered, a greater number of pupils would be expected to enrol in the private system. Then, the total amount saved by the Government is $a - s$ for each pupil multiplied by $g(s)$, the number of pupils in the private system. So, the total amount saved with the subsidy s is $S(s)$, where

$$S(s) = g(s)(a-s) \tag{5}$$

Note that $S(s) \geq 0$ and $S(a) = 0$. In (5), formally s can take any real value, but it is assumed that the subsidy offered by the Government lies in the interval $[0,a]$, for a negative subsidy would mean a tax on people sending their children to a private school, while a subsidy greater than a would mean the Government would be paying more for a pupil going to private school than to a state school. Equation (5) shows that the savings function S is comprised of two contradictory tendencies, for $g(s)$ increases as s increases, but $a - s$ decreases as s increases.

Savings will be maximised when the function S has a maximum over the interval $[0,a]$. A Government wishing to maximise savings without regard for anything else should try to find the level of subsidy where this maximum will occur.

The simplest choice for g is a linear function. We let N_0 be the number of pupils who enrol in the private system when the subsidy is zero; thus, $N_0 = g(0)$. The value of N_0 is a matter of controversy, with some letter writers to the *Sydney Morning Herald* (e.g., Davies, 2005) holding that it is in effect 0, while others (e.g., Heaton, 2005) hold that subsidies have little effect and that $g(s)$ is always close to N_0. Also, we define N'_0 by putting $N'_0 = g(a)$. Thus, N'_0 is the

number of private school pupils who would enrol under the maximum subsidy a and we assume that $N'_0 > N_0$. Thus, g is given in the linear case by

$$g(s) = N_0 + \left(\frac{N'_0 - N_0}{a} \right) s \qquad (6)$$

Definition
If $N_0 > 0$, make the definition that $\rho = \dfrac{N'_0}{N_0}$. If $N_0 = 0$, let $\rho = \infty$.

Thus, $\rho > 1$ and ρ measures the "sensitivity" of the "market" to subsidies.

For the time being, ρ is kept as a given constant value, and we shall think later about what happens as ρ varies. It follows from (5) and (6) and a routine calculation that the savings are given by

$$S(s) = \left(N_0 + \left(\frac{N'_0 - N_0}{a} \right) s \right)(a - s)$$

$$= \frac{N_0}{a}\big(a + (\rho - 1)s\big)(a - s) \qquad (7)$$

so that S is a quadratic function in s. Hence, we know that the point where S has an overall maximum is the midpoint of the zeros of S.
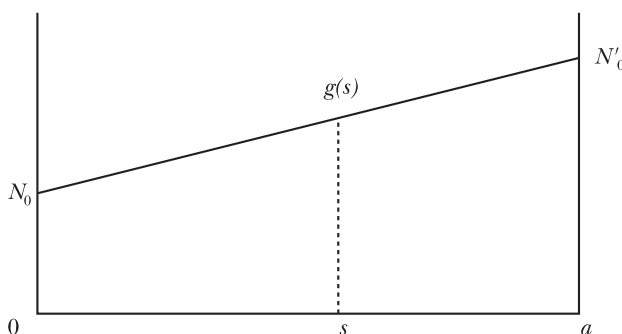

Figure 1. Let g(s) be the number of private school enrolments at the subsidy s per pupil.
The figure shows the graph of g where g is assumed linear.

We see from (7) that the zeros of S are at a and $\dfrac{-a}{\rho - 1}$, so S has an overall maximum value at the mid-point of these two zeros, namely it has a maximum at

$$\frac{a}{2}\left(\frac{\rho - 2}{\rho - 1} \right) \qquad (8)$$

Note that this point where the maximum occurs is negative if $\rho < 2$, is 0 if $\rho = 2$, and is in $(0, \frac{a}{2})$ if $\rho > 2$. As ρ increases, the point where the overall maximum occurs increases, changing from negative to positive at $\rho = 2$ (see Figures 2 and 3). However, we are interested in the maximum value of $S(s)$ for $0 \le s \le a$. We put

$$S_{\max} = \max\{S(s) : 0 \le s \le a\}$$

Thus, if $1 < \rho \le 2$, $S_{\max} = S(0) = aN_0$; while if $\rho > 2$, a routine calculation using (8) gives

$$S_{\max} = S\left(\frac{a}{2}\left(\frac{\rho - 2}{\rho - 1}\right)\right)$$

$$= \frac{N_0}{a}\left(a + (\rho - 1)\frac{a}{2}\left(\frac{\rho - 2}{\rho - 1}\right)\right)$$

$$= \frac{aN_0\rho^2}{4(\rho - 1)} \tag{9}$$

Now, we saw in (4) that the subsidy offered by the Government is

$$s = \frac{2}{3}a$$

so the savings under current Government policy are

$$S\left(\frac{2a}{3}\right)$$

and we call this value S_{gov}. We let θ be a given number in $[0,1]$, and first we calculate the difference between the maximum possible savings and the savings at the subsidy θa; that is, we calculate $S_{\max} - S(\theta a)$. There are two cases to consider: $1 < \rho \leq 2$ and $\rho > 2$.
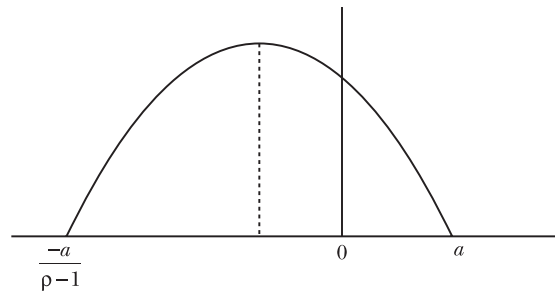

Figure 2. The graph is of the savings S against the subsidy s, in a case where $1 < \rho < 2$.
Note that the overall maximum value of the savings is at the negative value $s = \frac{a}{2}\left(\frac{\rho - 2}{\rho - 1}\right)$,
indicated by the dotted vertical line, while the maximum value of S(s) for $0 \leq s \leq a$ occurs when $s = 0$.
The graph illustrates that as ρ increases from 1 to 2, the subsidy at which the
maximum savings $S_{\max}$ occur remains the same at $s = 0$.
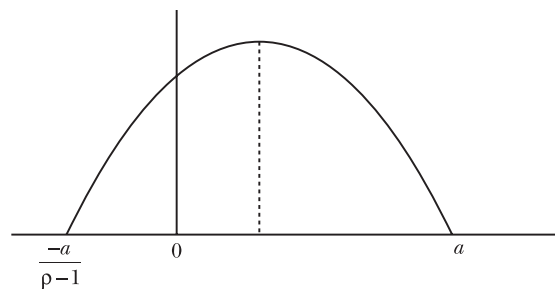

Figure 3. The graph is of the savings S against the subsidy s when $\rho \geq 2$. In this case, the overall
maximum savings are the same as the maximum savings $S_{\max}$ for $0 \leq s \leq a$, and occur when
$s = \frac{a}{2}\left(\frac{\rho - 2}{\rho - 1}\right)$, *indicated by the dotted vertical line, which is between 0 and* $\frac{a}{2}$. *Note that* $\frac{a}{2}$ *is half*
the maximum possible subsidy. Figures 2 and 3 together illus-
trate that as ρ increases from 1 to ∞, the subsidy at which the maximum savings $S_{\max}$ occur is at $s = 0$
for $1 < \rho \leq 2$, and then it increases to $s = \frac{a}{2}$ *as $\rho \to \infty$.*

We illustrate what happens in the case $\rho > 2$. A routine calculation based on (7), (8) and (9) gives

$$S_{\max} - S(\theta a) = S(\theta a)\left(\frac{S_{\max}}{S(\theta a)} - 1\right)$$

$$= S(\theta a)\left(\frac{\rho^2}{4(\rho-1)} \cdot \frac{1}{(1+(\rho-1)\theta)(1-\theta)} - 1\right) \qquad (10)$$

In particular, when $\theta = \frac{2}{3}$, equation (10) simplifies to

$$S_{\max} - S_{\text{gov}} = S_{\max} - S\left(\frac{2a}{3}\right) = S\left(\frac{2a}{3}\right) \cdot \frac{1}{4}\left(\frac{\rho^2 + 4\rho + 4}{2\rho^2 - \rho - 1}\right) \qquad (11)$$

A similar calculation in the case $1 < \rho \le 2$ gives

$$S_{\max} - S_{\text{gov}} = S_{\max} - S\left(\frac{2a}{3}\right) = S\left(\frac{2a}{3}\right) \cdot \left(\frac{8 - 2\rho}{2\rho + 1}\right) \qquad (12)$$

Now put

$$u(\rho) = \frac{8 - 2\rho}{2\rho + 1} \text{ for } 1 < \rho \le 2; \text{ and } u(\rho) = \frac{1}{4}\left(\frac{\rho^2 + 4\rho + 4}{2\rho^2 - \rho - 1}\right) \text{ for } 2 < \rho < \infty$$

Differentiating u with respect to ρ shows that u is decreasing and we see also that $u(\rho) \to \frac{1}{8}$ as $\rho \to \infty$. The following result is then immediate from (11) and (12).

Theorem
For all $1 < \rho < \infty$,

$$S_{\max} - S_{\text{gov}} = u(\rho)S_{\text{gov}} \ge \frac{1}{8}S_{\text{gov}}$$

The value of ρ which would give the impression that current Government policy is trying to save the maximum possible amount is when $S_{\max} - S_{\text{gov}}$ is a minimum: that is, when $\rho = \infty$, corresponding to $N_0 = 0$ and $u(\rho) = \frac{1}{8}$. According to the Minister, $S_{\text{gov}} = \$4$ billion. Thus, at the Government's current subsidy level, and according to the linear model, and assuming $\rho = \infty$, we have

$$S_{\max} - S_{\text{gov}} = \frac{1}{8}S_{\text{gov}} = \frac{1}{8} \times 4 \times 10^9 \text{ dollars} = \$500 \text{ million}$$

In fact, it follows from the theorem above that whatever the value of ρ,

$$S_{\max} - S_{\text{gov}} \ge \$500 \text{ million}$$

Thus, under the linear model, $500 million is the smallest amount *more* that can be saved compared with what the Government is currently saving. In fact, this amount is most likely greater, as the actual value of ρ is almost certainly comparatively small, which means that at the actual value of ρ, $S_{\max} - S_{\text{gov}}$ is likely to be substantially greater than $500 million. In fact, we can try and estimate ρ from the Minister's data: we let N be the total number of pupils in the combined state and private systems. We know from the Minister's figures that

$$\frac{8N}{25} = 0.32 N = g\left(\frac{2a}{3}\right)$$

so (6) gives

$$\frac{8}{25} N = N_0 + \frac{2}{3}\left(N'_0 - N_0\right) = \frac{N_0}{3} + \frac{2N'_0}{3}$$

Hence,

$$\rho = \frac{N'_0}{N_0} = \frac{3}{2}\left(\frac{8}{25}\frac{N}{N_0} - \frac{1}{3}\right) = \frac{12}{25}\left(\frac{N}{N_0}\right) - \frac{1}{2} \tag{13}$$

Although it seems difficult to estimate ρ, we may feel more comfortable in estimating

$$\frac{N_0}{N}$$

which is the ratio telling us the proportion of school pupils who would go to a private school even if there were no subsidy. For example, if we think that 20% of pupils would go to private schools even if there were no subsidy, (13) would give $\rho = 1.9$, and then the equation $S_{max} - S_{gov} = u(\rho) S_{gov}$ tells us that the Government could save \$3500 million more than it currently is, under the linear model. However, if we think that 10% of pupils would go to private schools even if there were no subsidy, the Government could save \$1252 million more than it currently is, under the linear model.

Limitations of the analysis and conclusions

If the function g is assumed non-linear the point where savings are maximised may be quite different from that in the linear case. However, the linear model is widely used in economics. The analysis takes no account of the differing circumstances between different schools, nor does the analysis take account of the splitting of school funding between state and federal Governments. The analysis is based solely on the three items of data given by the Minister. Even so, the analysis is strongly suggestive that the saving of public funds in this area is an incidental effect of policy, rather than its purpose. The analysis presented here is a particular approach to what is a special case of a supply and demand problem. Such problems occur widely wherever one is tying to optimise a quantity in the face of conflicting tendencies and, with appropriate changes, possibly they may be tackled by an adaptation of the techniques described in this paper. Three possible areas are: maximising profit in retailing, where there are conflicting tendencies between the price of an item and the number of items one can expect to sell at that price; taxation policy, where there are conflicting tendencies between the amount of income tax collected and the incentive to work (this is related to the "Laffer curve" which featured in tax policy under US President Ronald Reagan); and university enrolments, where there are conflicting tendencies between the cost to a student of enrolling and the number of students who enrol. Further details of the analysis and material related to school mathematics may be found on the author's website (www.uow.edu.au/~nillsen).

References

Davies, J. (29 March 2005). Letter. *The Sydney Morning Herald.*

Gittins, Ross (23 March 2005). Public money leaks away to the better off. *The Sydney Morning Herald.*

Heaton, Grant (28 March 2005). Letter. *The Sydney Morning Herald.*

Nelson, Brendan (25–27 March 2005). Letter. *The Sydney Morning Herald.*